Engineering Notes

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A Proposed Scheme for Maneuver-Dependent Control of Stability

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Introduction

A MANEUVER-DEPENDENT stability control system is proposed. The system is essentially an algorithm which computes the optimal output controls that satisfy predetermined stability requirements for aircraft performing maneuver due to input pilot command. Linear filter theory together with a model following technique are employed. The optimal control vector is obtained by minimizing a quadratic performance index in the state vector estimate error and the control vector (minimum control effort). A first-order perturbation approach is used to change the eigenvalues of the model to the desired ones by changing the elements of the model coefficient matrix (stability derivatives), thus controlling the aircraft stability.

Filtering and Model Following

The linearized equations describing the continuous state of the aircraft are

$$\dot{x} = Ax + Bu + Cw \qquad y = Mx + v \tag{1}$$

where

x = the state vector of dimension n
u = the control vector of dimension m
w = the process noise vector of dimension l
y = the measurement vector of dimension p
v = the measurement noise vector of dimension p
A, B, C, and M are constant coefficient matrices of dimensions $(n\times n)$, $(n\times m)$, $(n\times l)$, and $(p\times n)$, respectively.

The objective is to find out the optimal control vector in such a way that the response of the actual dynamic system (aircraft) follows the response of a model system defined in the continuous state by

$$\dot{x}_m = A_m x_m + B_m \delta_c \qquad \dot{\delta}_c = A_c \delta_c \tag{2}$$

where the subscript m denotes the model system, the subscript c denotes command input, and δ_c is the command input vector of dimension q. The actual and model state vectors are of the same dimension. It is the discrete forms of the state and model equations that we are interested in rather than the continuous forms. Equations (1) and (2) may thus be written in discrete forms as[1]

$$x_{k+1} = \phi_k(\Delta t)x_k + L_k(\Delta t)u_k + G_k(\Delta t)w_k \tag{3a}$$

$$y_k = M_k x_k + v_k \tag{3b}$$

and

$$x_{m_{k+1}} = \Psi_k(\Delta t)x_{m_k} + H_k(\Delta t)\delta_{c_k} \tag{4a}$$

$$\delta_{c_{k+1}} = \chi_k(\Delta t)\delta_{c_k} \tag{4b}$$

where Δt is the sample time interval $(t_{k+1}-t_k)$; $\phi_k(\Delta t)$, $\Psi_k(\Delta t)$, and $\chi_k(\Delta t)$ are the transition matrices for the actual state, the model, and the command input equations, respectively. The matrices $L_k(\Delta t)$, $G_k(\Delta t)$, and $H_k(\Delta t)$ are given by

$$L_k(\Delta t)u_k = \int_k^{k+1}\phi(t_{k+1},\tau)B(\tau)u(\tau)\mathrm{d}\tau \tag{5a}$$

$$G_k(\Delta t)w_k = \int_k^{k+1}\phi(t_{k+1},\tau)C(\tau)w(\tau)\mathrm{d}\tau \tag{5b}$$

$$H_k(\Delta t)\delta_{c_k} = \int_k^{K+1}\Psi(t_{k+1},\tau)B_m(\tau)\delta_c(\tau)\mathrm{d}\tau \tag{5c}$$

The performance index most suited for the present problem is a penalized quadratic difference between the actual state and the model state together with a penalized quadratic control effort, i.e.

$$J_k = (\hat{x}_k - x_{m_k})^T S_k(\hat{x}_k - x_{m_k}) + u_k^T U_k u_k \tag{6}$$

where $\hat{x}_k$ is the optimal estimate of the state vector based on the obtained measurement vector y; and S_k, U_k are positive definite symmetric weighting matrices.

The model following problem is essentially composed of two problems to be solved in sequence. The first one is a filtering problem where the best estimate of the actual state variables are to be determined from noisy measurements. The second problem is an optimal control problem where the performance index given by Eq. (6) is to be minimized.[2,3]

Following the discrete Kalman filter approach, for zero mean white process noise w_k with covariance Q_k and zero mean measurement noise v_k with covariance R_k, the unbiased optimal linear estimator takes the form

$$\hat{x}_k(+) = (I - K_k M_k)\hat{x}_k(-) + K_k y_k \tag{7}$$

where I is the identity matrix, and K_k is the Kalman gain matrix given by

$$K_k = P_k(-)M_k^T[M_k P_k(-)M_k^T + R_k]^{-1} \tag{8}$$

$P_k(-)$ is the error covariance matrix of the estimate $\hat{x}_k(-)$ and is given by $E[\tilde{x}_k(-)\ \tilde{x}_k^T(-)]$, where $\tilde{x}_k(-)$ is the difference (error) between the estimated state vector $\hat{x}_k(-)$ and the actual state vector $x_k(-)$.

The preceding expression for the Kalman gain matrix is based on the assumption that both process noise and measurement noise are uncorrelated. The unbiased linear estimator given by Eq. (7) describes the updating of the state vector estimate at a particular interval of time t_k. The extrapolation of the state vector estimate along successive time intervals could be obtained from Eq. (3a) as

$$\hat{x}_k(-) = \phi_{k-1}\hat{x}_{k-1}(+) + L_{k-1}u_{k-1}(+) + G_{k-1}w_{k-1}(+) \tag{9}$$

with error covariance matrix in extrapolated form:

$$P_k(-) = \phi_{k-1}P_{k-1}(+)\phi_{k-1}^T + L_{k-1}R_{k-1}L_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T \tag{10}$$

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Substituting the state estimate extrapolation expression Eq. (9) into the estimator form Eq. (7), we get

$$\hat{x}_k(+) = (I - K_k M_k)\phi_{k-1}x_{k-1}(+) + (I - K_k M_k) \times [L_{k-1}u_{k-1}(+) + G_{k-1}w_{k-1}(+)] + K_k y_k \quad (11)$$

Equation (11) gives the extrapolation expression of the state vector estimate of the system discrete form given by Eq. (3). Let us define the model following error vector by ϵ_k:

$$\epsilon_k = \hat{x}_k(+) - x_{m_k}(+) \quad (12)$$

Subtracting Eq. (4a) from Eq. (11) and making use of Eq. (12) we get, after rearranging,

$$\epsilon_k = \Psi_{k-1}\epsilon_{k-1} + V_{k-1}\hat{x}_{k-1}(+) + T_k L_{k-1}u_{k-1}(+) + W_{k-1} + K_k y_k - H_{k-1}\delta_{c_{k-1}} \quad (13)$$

where

$$T_k = I - K_k M_k \quad (14a)$$

$$V_{k-1} = T_k\phi_{k-1} - \Psi_{k-1} \quad (14b)$$

$$W_{k-1} = T_k G_{k-1} w_{k-1}(+) \quad (14c)$$

The performance index, Eq. (6), may be written as

$$J_k = \epsilon_k^T S_k \epsilon_k + u_k^T U_k u_k \quad (15)$$

The optimal control vector can be obtained by ordinary calculus; i.e., taking $\partial J_k/\partial u_k = 0$ after some manipulations we get

$$u_k^* = [L_{k-1}^T T_k^T S_k T_k L_{k-1} + U_k]^{-1}(L_{k-1}^T T_k^T S_k)[\Psi_{k-1}\epsilon_{k-1} + V_{k-1}\hat{x}_{k-1}(+) + W_{k-1} + K_k y_k - H_{k-1}\delta_{c_{k-1}}] \quad (16)$$

The actual and model systems are both stationary, therefore their transition matrices are given by

$$\phi_k = e^{A\Delta t} \qquad \Psi_k = e^{A_m \Delta t} \quad (17)$$

Δt is the sample time interval $(t_{k+1} - t_k)$.

Stability Control

The stability control (augmentation) technique is essentially a model following technique with stability and handling requirements embodied in the model equations. Since the eigenvalues of the coefficient matrix determine the stability status of the system, therefore, controlling the eigenvalues, i.e. changing them to the desired values, leads to stability control. The only way to change the eigenvalues is to change some or all of the elements of the coefficient matrix. In the stability augmentation sense, the model state is the one that describes the desired performance. This suggests that the model coefficient matrix A_m be written in the modified form

$$A_m = A_m^0 + \sum_{s=1}^{n} (\Delta A_m)_s \quad (18)$$

where A_m^0 is the standard coefficient matrix (with no stability control), and $(\Delta A_m)_s$ is a controllable differential $(n \times n)$ matrix whose elements Δa_{ij} $(i,j = 1,\ldots,n)$ are functions of $\Delta\lambda_s$ $(s = 1,\ldots,n)$. $\Delta\lambda_s$ is the difference between the desired eigenvalue λ_s and the standard one λ_s^0.

For the model state equation, the eigenvalues of their coefficient matrix are the roots of the equation

$$\det(\lambda I - A_m) = 0 \quad (19)$$

which is an nth-order algebraic equation in λ.

Equation (19) may be written more conveniently in polynomial form as

$$\alpha^T\omega = 0 \quad (20)$$

where the vectors α and ω are defined by

$$\alpha^T = [1 \ \alpha_{n-1} \ \alpha_{n-2} \cdots \alpha_1 \ \alpha_0]$$

$$\omega^T = [\lambda^n \ \lambda^{n-1} \ \lambda^{n-2} \cdots \lambda \ 1]$$

The vector α is a function of the elements of the model coefficient matrix A_m. Any of the matrix eigenvalues will be changed from its standard value $\lambda^{(0)}$ to the modified value $\lambda = \lambda^{(0)} + \Delta\lambda$ when any or all of the elements of A_m change from a_{ij} to $a_{ij} = a_{ij}^{(0)} + \Delta a_{ij}$ $(i,j = 1,\ldots,n)$. Taking the first-order variation of Eq. (20), we get, in expanded form,

$$[\lambda^{(0)n} + n\lambda^{(0)n-1}\Delta\lambda] + [\alpha_{n-1}^0 + (\partial\alpha_{n-1}/\partial a_{ij})\Delta a_{ij}] \times [\lambda^{(0)n-1} + (n-1)\lambda^{(0)n-2}\Delta\lambda] + [\alpha_{n-2}^0 + (\partial\alpha_{n-2}/\partial a_{ij})\Delta a_{ij}] \times [\lambda^{(0)n-2} + (n-2)\lambda^{(0)n-3}\Delta\lambda] + \ldots + [\alpha_1^0 + (\partial\alpha_1/\partial a_{ij})\Delta a_{ij}] \times [\lambda^{(0)} + \Delta\lambda] + [\alpha_0^0(\partial\alpha_0/\partial a_{ij})\Delta a_{ij}] = 0 \quad (21)$$

Since the reference (standard) values satisfy Eq. (20), Eq. (21) reduces to

$$\{n\lambda^{(0)n-1} + [\alpha_{n-1}^0 + (\partial\alpha_{n-1}/\partial a_{ij})\Delta a_{ij}](n-1)\lambda^{(0)n-2} \times [\alpha_{n-2}^0 + (\partial\alpha_{n-2}/\partial a_{ij})\Delta a_{ij}](n-2)\lambda^{(0)n-3} + \ldots + [\alpha_1^0 + (\partial\alpha_1/\partial a_{ij})\Delta a_{ij}]\}\Delta\lambda + \lambda^{(0)n-1}(\partial\alpha_{n-1}/\partial a_{ij})\Delta a_{ij} + \lambda^{(0)n-2}(\partial\alpha_{n-2}/\partial a_{ij})\Delta a_{ij} + \ldots + \lambda^{(0)}(\partial\alpha_1/\partial a_{ij})\Delta a_{ij} + (\partial\alpha_0/\partial a_{ij})\Delta a_{ij} = 0 \qquad (i,j = 1,\ldots,n) \quad (22)$$

Solving Eq. (22) for Δa_{ij} we get

$$\Delta a_{ij} = -\Delta\lambda[k/(\beta_{ij}\Delta\lambda + \theta_{ij})] \qquad (i,j = 1,\ldots,n) \quad (23a)$$

$$\beta_{ij} = (n-1)\lambda^{(0)n-2}(\partial\alpha_{n-1}/\partial a_{ij}) + (n-2)\lambda^{(0)n-3}(\partial\alpha_{n-2}/\partial a_{ij}) + \ldots + (\partial\alpha_1/\partial a_{ij}) \quad (23b)$$

$$\theta_{ij} = \lambda^{(0)n-1}(\partial\alpha_{n-1}/\partial a_{ij}) + \lambda^{(0)n-2}(\partial\alpha_{n-2}/\partial a_{ij}) + \ldots + \lambda^{(0)}(\partial\alpha_1/\partial a_{ij}) + (\partial\alpha_0/\partial a_{ij}) \qquad (i,j = 1,\ldots n) \quad (23c)$$

$$K = n\lambda^{(0)n-1} + (n-1)\lambda^{(0)n-2}\alpha_{n-1}^0 + (n-2)\lambda^{(0)n-3}\alpha_{n-2}^0 + \ldots + \alpha_1^0 \quad (23d)$$

The system eigenvalues may be real or complex, but a_{ij}, Δa_{ij}, and α are real. Separating real and imaginary parts of Eq. (21) we get after some manipulations,

$$\Delta a_{ij} = -\nu_{ij}/\mu_{ij} \quad (24a)$$

$$\nu_{ij} = (\Delta\lambda_R K_R - \Delta\lambda_I K_I)\epsilon_1 + (\Delta\lambda_I K_R + \Delta\lambda_R K_I)\epsilon_2 \quad (24b)$$

$$\mu_{ij} = \epsilon_1^2 + \epsilon_2^2 \quad (24c)$$

$$\epsilon_1 = (\beta_{ij})_R\Delta\lambda_R - (\beta_{ij})_I\Delta\lambda_I + \theta_R \quad (24d)$$

$$\epsilon_2 = (\beta_{ij})_I\Delta\lambda_R + (\beta_{ij})_R\Delta\lambda_I + \theta_I \quad (24e)$$

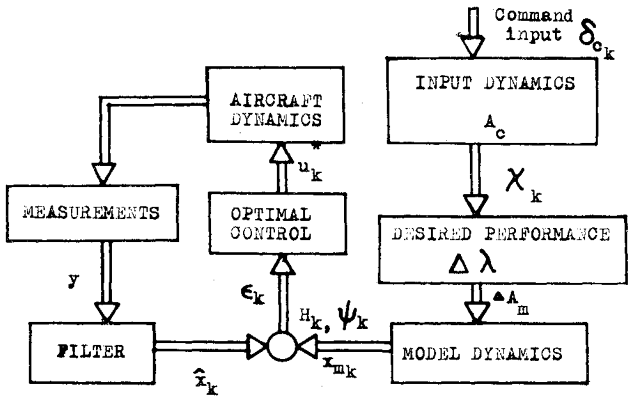

Fig. 1 Stability-controlled control loop.

where subscripts R and I denote real and imaginary parts, respectively.

For a particular aircraft the choice of the change in one of the model eigenvalues is based on its performance capabilities. Once $\Delta\lambda$ is specified, the elements of the differential matrix ΔA_m could be computed from Eq. (24).

The model transition matrix that should be used in the expression for the optimal control vector, Eq. (16), is thus

$$\Psi_k = \exp\left[A_m + \sum_{s=1}^{n} (\Delta A_m)_s\right]\Delta t$$

A block diagram for the proposed control loop is shown in Fig. 1.

References

[1] Gelb, A. (ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, Mass., 1974.

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Technical Comments

Comment on "Simplified Methods of Predicting Aircraft Rolling Moments Due to Vortex Encounters"

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THE simple formula derived by Barrows[1] relating the aerodynamic rolling moment according to strip theory to the rolling moment according to lifting-line theory for a wing of elliptical planform subjected to an arbitrary antisymmetric angle-of-attack distribution is a long-established and well-known result in wing theory.[2,3] Extensive applications of this formula to stability and control and to aeroelasticity have been given in the literature.[4-7] There is a similar formula[2,3] for the lift on an elliptic wing due to an arbitrary symmetric angle-of-attack distribution. These special results for the elliptic wing follow from the fact that for this planform all the coefficients of the Fourier series for spanwise circulation distribution in lifting-line theory may be determined explicitly and exactly by quadrature without solving simultaneous equations (which in principle are infinite in number). As is well known in lifting-line theory, for all planforms the total lift depends only on the first Fourier coefficient, whereas the total rolling moment depends only on the second coefficient.

The approximate formulas for rolling moments of linearly tapered wings in Ref. 1 correspond to results originally obtained by Victory and quoted in Ref. 7. As shown in Victory's results (and indicated in Barrow's work as well), the relation between rolling moment from strip theory M_r and the rolling moment M_r' according to lifting-line theory is

$$\frac{M_r'}{M_r} = \frac{1}{1+(c/\text{Æ})} \tag{1}$$

where Æ is the aspect ratio and the exact value of c is 4 for the elliptic wing in all cases of antisymmetric loading. For other wing planforms, c is a function of both the planform and the loading so the formula is of much less general utility for these cases. For example, for linear antisymmetric angle-of-attack distributions, c varies from 3.15 to 6.03 as the wing taper ratio λ (equal to the ratio of tip chord to root chord) varies from 0 to 1.0, whereas for cubic antisymmetric twist, c varies from 2.13 to 7.65 as λ varies from 0 to 1.0. These values indicate that over the full range of λ, differences in loading may alter c substantially.

There is, in fact, a development of lifting-line theory by Sears[7] in which the solution of the problem for any wing planform is made mathematically equivalent to the solution for the elliptic wing in that the spanwise circulation distribution may be given explicitly as a series of terms, each of which can be expressed independently of the others so that simultaneous equations need not be solved for each new distribution of imposed angle of attack. For tapered planforms, the Sears solution is of form

$$M_r' = qSb\sum_{n=1}^{\infty} m_n \frac{\int_0^{\pi}\alpha(\tau)\varphi_n(\tau)\mathrm{d}\tau}{1+(c_n/\text{Æ})} \tag{2}$$

where q is the dynamic pressure, S is the wing area, α is the imposed angle of attack, and the angle τ is related to the spanwise coordinate y by $y = b/2\cos\tau$, b being the wing span. The functions $\varphi_n(\tau)$, of which there is a denumerably infinite set for any given wing-taper ratio, are eigenfunctions of the homogeneous linear integral equation obtained by omitting the term in geometric angle of attack from the integral equation of lifting-line theory. c_n increases monotomically as n increases and is, to within a multiplicative constant depending on taper ratio, the negative of the eigenvalues corresponding to φ_n. m_n is effectively a rolling-moment coefficient corresponding to circulation distributed proportional to $\varphi_n(\tau)$. [In fact, in analogy to the elliptic case in which the spanwise circulation $\Gamma(\tau) = 2bU_0\Sigma A_n \sin n\tau$, for the more general case $\Gamma = 2bU_0\Sigma a_n\varphi_n(\tau)$.]

For wings with linear taper in planform, Sears has given numerical values for a representative set of taper ratios; thus c_n and m_n may readily be found. Unlike Eq. (1), Eq. (2) admits no simple relationship between the value of rolling moment obtained from strip theory and the value from lifting-line theory. Moreover, if $\alpha(\tau)$ is orthogonal to all values of φ_n except, say, φ_m, then the summation contains only a single term, the mth, and the value of c_m in the denominator

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